ADVANCED RESEARCH & INNOVATIONS
International Journal of Advanced Research & Innovations
Volume 15, Issue-2
September, 2025, P74
ISSN: 2319-9253

International Conference on Emerging Trends in Engineering, Technology & Management (ICETM-2025)
Conducted by *Viswam Engineering College (UGC—Autonomous Institution)* held on 11th & 12th, April- 2025

THE SIGNLESS LAPLACIAN MATRIX AND ENERGY OF HESITANCY FUZZY GRAPHS WITH NEW BOUNDS

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Abstract : This research investigates the Sign less Laplacian adjacency matrix and the Sign less Laplacian energy of hesitancy fuzzy graphs. This research focusses on the sign less Laplacian matrix, which is useful for understanding the structure and properties of these graphs. This research derives an upper and lower bound for the sign less Laplacian energy using known graph parameters such as maximum degree, and eigenvalues. These bounds help estimate the sign less Laplacian energy more easily without needing full calculations. Furthermore, an illustrative example is provided to demonstrate the application of these results.
Keywords: *Fuzzy graph, hesitancy fuzzy graph, Laplacian adjacency matrix, Sign less Laplacian adjacency matrix, Eigenvalues, Lower and upper bounds.*

1. INTRODUCTION

In 1965, Zadeh introduced the concept of fuzzy sets (FSs) and fuzzy relations. Kauffman introduced the fuzzy graph (FG) idea in 1973 and Rosenfeld discussed the concept of FG in 1975. Rosenfeld also considered fuzzy relations between fuzzy sets and developed the notion of fuzzy graphs, obtaining analogues of various graph notions. Bhattacharya shows how fuzzy groups can be connected to fuzzy graphs in a natural manner. The notions of eccentricity and center are introduced as well as some properties of fuzzy graphs. FS theory is one expansion and generalisation of this theory that is called the hesitation fuzzy set (HFS). HFSs give all the possible degrees of membership that are independent from one another, as opposed to fuzzy sets which give the degree of membership. A distinct approach to HFS in graph theory and decision making from the one presented in. In the most real-world problems are extremely complex and contain vague information. To test the lack of certainty, Torra has initiated future development of the FSs and called it Hesitant Fuzzy Sets (HFSs). HFSs are encouraged to deal with the prevalent problem that appears to be in fixing the degree of the membership of the element from many possible values. The researcher had to find methods and techniques to solve a problem and find solutions in. Pathinathan, Jon Arockiaraj, and Jesintha Rosline described the Hesitancy Fuzzy Graph (HFG) and provided examples. Related results were also studied and demonstrated. Various properties were discussed in. HFGs are applied to capture the underlying complexity of the choice of membership degree for an element that makes one hesitate.

Gutman and Zhou introduced the concept of the Laplacian energy (LE) of a graph. The result for Laplacian energy and the energy of a graph G, and the results for the sum of the chromatic numbers of a graph G and its complement G, were shown by Nordhaus and Gaddum. Rahimi and Fayazi introduced the concept of LE

of FGs and the boundaries on LE of FGs are provided. The LE of an IFG is generalised from the concept of LE of FGs by Sharief Basha and Kartheek. Pirzada and Saleem khan introduced the concept of signless Laplacian adjacency matrix and energy of a graph. They are also obtained lower and upper bounds with suitable example. This idea has been extended to fuzzy graphs, where edge memberships are not binary but instead take values in the interval [0,1], allowing better modeling of uncertain relationships. Moreover, the notion of signless Laplacian energy has also been applied to intuitionistic fuzzy graphs (IFGs), where both membership and non-membership degrees are considered, offering a more expressive model of uncertainty.
This paper is organized as follows: Section Two discusses the concept of hesitancy fuzzy graphs (HFGs), the signless Laplacian energy of HFGs, and provides some basic definitions along with a suitable example illustrating the concept. Section Three presents the upper and lower bounds of the signless Laplacian energy of HFGs. Section Four concludes the paper with a summary of the results.

2. PRELIMINARIES

In this section, we present the fundamental concepts and definitions that are essential for understanding the discussions and results presented in this paper.

2.1. Hesitancy Fuzzy Graph

A hesitancy fuzzy graph is a generalization of a fuzzy graph that accounts for situations where there is uncertainty or hesitation in the relationship between vertices. A fuzzy graph allows each edge to have a membership value between 0 and 1, representing the strength of the connection. Hesitancy fuzzy graphs extend this by introducing the concept of hesitation, which represents the uncertainty in the relationships between vertices.

Definition 2.1.1: A HFG is of the form $HG = (V, E, \mu, \gamma, \beta)$,where

(a) $V = \{v_1, v_2, v_3 \dots v_p\}$ such that $\mu_1 : V \to [0,1]$, $\gamma_1 : V \to [0,1]$ and $\beta_1 : V \to [0,1]$are denotes the degree of membership, nonmembership and hesitancy of the element $v_i \in V$ and $\mu_1(v_i) + \gamma_1(v_i) + \beta_1(v_i) = 1$ for every $v_i \in V$, where

$$\beta_1(v_i) = 1 - [\mu_1(v_i) + \gamma_1(v_i)] \quad \to (1)$$

(b) $E \subseteq V \times V$ where $\mu_2 : V \times V \to [0,1]$, $\gamma_2 : V \times V \to [0,1]$ and $\beta_2 : V \times V \to [0,1]$ are such that,

$$\mu_2(v_i, v_j) \le min\,[\mu_1(v_i), \mu_1(v_j)] \quad \to (2)$$

$$\gamma_2(v_i, v_j) \le max\,[\gamma_1(v_i), \gamma_1(v_j)] \quad \to (3)$$

$$\beta_2(v_i, v_j) \le min\,[\beta_1(v_i), \beta_1(v_j)]\ and \quad \to (4)$$

$$0 \le \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) + \beta_2(v_i, v_j) \le 1 \quad \to (5)$$

for every$(v_i, v_j) \in E$.

Note that, here $V$and E is the set of vertices and edges, μ, γ and β is the membership function, non-membership function and hesitant function defined on$V \times V$. We define μ, γ and β by$\mu(v_i, v_j), \gamma(v_i, v_j)$ and$\beta(v_i, v_j)$.

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Example:

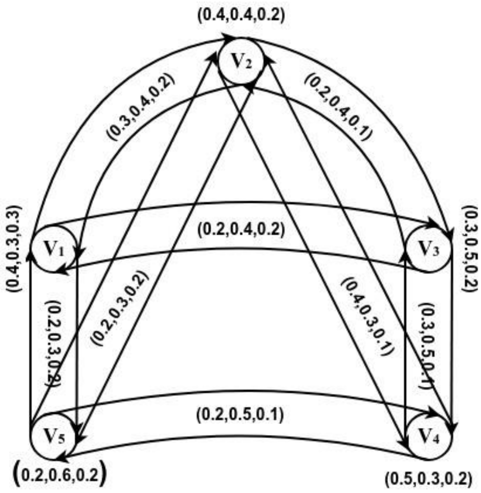

Figure: Hesitancy Fuzzy Graph

Definition2.1.2. Consider a hesitancy fuzzy graph$HG = (V, E, \mu, \gamma, \beta)$, then the adjacency matrix of HFG is defined as $A(HG) = [a_{ij}]$ where $a_{ij} = (\mu_{ij}, \gamma_{ij}, \beta_{ij})$, here μ_{ij} represents the strength of the relationship between v_i and v_j, γ_{ij} represents the strength of the non-relationship between v_i and v_j and β_{ij}represents the strength of the relationship between v_i and v_j.

The adjacency matrix of hesitancy fuzzy Graph is defined from the above figure1.

$$A(HG) = \begin{bmatrix} (0,0,0) & (0.3,0.4,0.2) & (0.2,0.4,0.2) & (0,0,0) & (0.2,0.3,0.2) \\ (0.3,0.4,0.2) & (0,0,0) & (0.2,0.4,0.1) & (0.4,0.3,0.1) & (0.2,0.3,0.2) \\ (0.2,0.4,0.2) & (0.2,0.4,0.1) & (0,0,0) & (0.3,0.5,0.1) & (0,0,0) \\ (0,0,0) & (0.4,0.3,0.1) & (0.3,0.5,0.1) & (0,0,0) & (0.2,0.5,0.1) \\ (0.2,0.3,0.2) & (0.2,0.3,0.2) & (0,0,0) & (0.2,0.5,0.1) & (0,0,0) \end{bmatrix}$$

Definition 2.1.3. The adjacency matrix $A(HG)$ of an HFG can be written as three matrices, one containing the entries as the membership function, other one containing the entries as the non-membership function, and last one containing the entries as the hesitancy function is defined as

$$i.e., A(HG) = [A_\mu(HG),\ A_\gamma(HG),\ A_\beta(HG)].$$

Where $A_\mu(HG)$ is the adjacency matrix of membership function, $A_\gamma(HG)$ is the is the adjacency matrix of nonmembership function and $A_\beta(HG)$ is the is the adjacency matrix of hesitant function

$$A_\mu(HG) = \begin{bmatrix} 0 & 0.3 & 0.2 & 0 & 0.2 \\ 0.3 & 0 & 0.2 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0 & 0.3 & 0 \\ 0 & 0.4 & 0.3 & 0 & 0.2 \\ 0.2 & 0.2 & 0 & 0.2 & 0 \end{bmatrix},\ A_\gamma(HG) = \begin{bmatrix} 0 & 0.4 & 0.4 & 0 & 0.3 \\ 0.4 & 0 & 0.4 & 0.3 & 0.3 \\ 0.4 & 0.4 & 0 & 0.5 & 0 \\ 0 & 0.3 & 0.5 & 0 & 0.5 \\ 0.3 & 0.3 & 0 & 0.5 & 0 \end{bmatrix} \text{ and}$$

$$A_\beta(HG) = \begin{bmatrix} 0 & 0.2 & 0.2 & 0 & 0.2 \\ 0.2 & 0 & 0.1 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0 & 0.1 & 0 \\ 0 & 0.1 & 0.1 & 0 & 0.1 \\ 0.2 & 0.2 & 0 & 0.1 & 0 \end{bmatrix}.$$

Definition2.1.5: Let $A(HG)$ be an adjacency matrix and $D(HG) = [d_{ij}]$ be a degree matrix of $HG = (V, E, \mu, \gamma)$ then $S_L(HG) = D(HG) + A(HG)$ is defined as the signless Laplacian adjacency matrix of HFG. The signless Laplacian adjacency matrix (SLAM) of an HFG can be written as three matrices one containing the entries as membership function, another one containing nonmembership function and last one containing the entries as hesitant function

$$i.e., S_L(A(HG)) = \left[S_L\left(A_\mu(HG)\right), S_L\left(A_\gamma(HG)\right), S_L\left(A_\beta(HG)\right)\right].$$

Where $S_L\left(A_\mu(HG)\right)$is the SLAM of membership function of HFG is, $S_L(A_\gamma(HG))$ is the SLAM of nonmembership function of HFG, and $S_L(A_\beta(HG))$ is the SLAM of hesitant function of HFG.

Now the SLAM of membership function of HFG is

$$S_L(A_\mu(HG)) = \begin{bmatrix} 0.7 & 0.3 & 0.2 & 0 & 0.2 \\ 0.3 & 1.1 & 0.2 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.7 & 0.3 & 0 \\ 0 & 0.4 & 0.3 & 0.9 & 0.2 \\ 0.2 & 0.2 & 0 & 0.2 & 0.6 \end{bmatrix}$$

The SLAM of nonmembership function of HFG is

$$S_L(A_\gamma(HG)) = \begin{bmatrix} 1.1 & 0.4 & 0.4 & 0 & 0.3 \\ 0.4 & 1.4 & 0.4 & 0.3 & 0.3 \\ 0.4 & 0.4 & 1.3 & 0.5 & 0 \\ 0 & 0.3 & 0.5 & 1.3 & 0.5 \\ 0.3 & 0.3 & 0 & 0.5 & 1.1 \end{bmatrix} \text{ and}$$

The SLAM of hesitant function of HFG is

$$S_L(A_\beta(HG)) = \begin{bmatrix} 0.6 & 0.2 & 0.2 & 0 & 0.2 \\ 0.2 & 0.6 & 0.1 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0.4 & 0.1 & 0 \\ 0 & 0.1 & 0.1 & 0.3 & 0.1 \\ 0.2 & 0.2 & 0 & 0.1 & 0.5 \end{bmatrix}$$

Definition 2.1.6: Let $S_L(A(HG))$ be a SLAM of membership function of HFG $HG = (V, E, \mu, \gamma, \beta)$. The Laplacian polynomial of HFG is the characteristic polynomial of its SLAM is defined as

$$\emptyset(HG, \mu) = \left|L\left(A(HG)\right) + \mu\, I_n\right|.$$

The roots of $\emptyset(HG, \mu)$ are the hesitancy fuzzy Laplacian eigenvalues of HFG HG.

(i) The SLAM of membership function of HFG

The SLAM of membership function of HFG is

$$S_L\left(A_\mu(HG)\right) = D(HG) + A_\mu(HG)$$

By using the SLAM of membership function, the degree of the matrix $D(HG)$ is defined as follows

$$D(HG) = \begin{bmatrix} 0.7 & 0 & 0 & 0 & 0 \\ 0 & 1.1 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 0 & 0.6 \end{bmatrix}$$

Now we have calculated the SLAM$S_L\left(A_\mu(HG)\right)$by using the degree of the matrix and the SLAMof membership function, we get

$$S_L(A_\mu(HG)) = \begin{bmatrix} 0.7 & 0.3 & 0.2 & 0 & 0.2 \\ 0.3 & 1.1 & 0.2 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.7 & 0.3 & 0 \\ 0 & 0.4 & 0.3 & 0.9 & 0.2 \\ 0.2 & 0.2 & 0 & 0.2 & 0.6 \end{bmatrix}$$

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We calculate the eigenvalues of the SLAM of membership function of HFG by solving its characteristic equation is

$\lambda_1 = 0.2466, \ \lambda_2 = 0.5664, \ \lambda_3 = 0.6598, \ \lambda_4 = 0.8000$, and $\lambda_5 = 1.7271$

The Laplacian energy of SLAM $S_L\left(A_\mu(HG)\right)$of membership function of HFG is

$$S_{LE}\left(A_\mu(HG)\right) = \sum_{i=1}^{n}\left|\lambda_i - \frac{2\sum_{1\le i\le j\le n}\mu(v_i,v_j)}{n}\right|$$

$$= \sum_{i=1}^{5}\left|\lambda_i - \frac{2(4.0)}{5}\right|$$

$$S_{LE}\left(A_\mu(HG)\right) = |\lambda_1 - 0.8000| + |\lambda_2 - 0.8000| + |\lambda_3 - 0.8000| + |\lambda_4 - 0.8000|$$
$$+|\lambda_5 - 0.8000|$$

Substituting $\lambda_1, \lambda_2, \lambda_3, \lambda_4$and $\lambda_5$values in above equation and solving we get

$$S_{LE}\left(A_\mu(HG)\right) = 1.8543$$

The signless Laplacian energy(S_{LE}) of the $S_{LE}\left(A_\mu(HG)\right)$is 1.8543

(ii) The signless Laplacian energy of SLAM of nonmembership function

$$A_\gamma(HG) = \begin{bmatrix} 0 & 0.4 & 0.4 & 0 & 0.3 \\ 0.4 & 0 & 0.4 & 0.3 & 0.3 \\ 0.4 & 0.4 & 0 & 0.5 & 0 \\ 0 & 0.3 & 0.5 & 0 & 0.5 \\ 0.3 & 0.3 & 0 & 0.5 & 0 \end{bmatrix}$$

The SLAMof nonmembership function of HFG is

$$S_L\left(A_\gamma(HG)\right) = D_\gamma(HG) + A_\gamma(HG)$$

By using the SLAMof nonmembership function, the degree of the matrix$D_\gamma(HG)$ is defined as follows

$$D_\gamma(HG) = \begin{bmatrix} 1.1 & 0 & 0 & 0 & 0 \\ 0 & 1.4 & 0 & 0 & 0 \\ 0 & 0 & 1.3 & 0 & 0 \\ 0 & 0 & 0 & 1.3 & 0 \\ 0 & 0 & 0 & 0 & 1.1 \end{bmatrix}$$

Substituting $D_\gamma(HG)$ and $A_\gamma(HG)$ in the equation of SLAM of nonmembership function of HFG, we get

$$S_L\left(A_\gamma(HG)\right) = \begin{bmatrix} 1.1 & 0.4 & 0.4 & 0 & 0.3 \\ 0.4 & 1.4 & 0.4 & 0.3 & 0.3 \\ 0.4 & 0.4 & 1.3 & 0.5 & 0 \\ 0 & 0.3 & 0.5 & 1.3 & 0.5 \\ 0.3 & 0.3 & 0 & 0.5 & 1.1 \end{bmatrix}$$

We calculate the eigenvalues of the SLAM of nonmembership function $L\left(A_\gamma(HG)\right)$ by solving its characteristic equation is

$\lambda_1 = 0.3468, \lambda_2 = 0.8761, \lambda_3 = 1.1634, \lambda_4 = 1.2960, \lambda_5 = 2.5178$

The signless Laplacian energy of SLAM$S_{LE}\left(A_\gamma(HG)\right)$ of membership function of HFG

$$S_{LE}\left(A_\gamma(HG)\right) = \sum_{i=1}^{n}\left|\lambda_i - \frac{2\sum\gamma(v_i,v_j)}{n}\right|$$

$$= \sum_{i=1}^{5} \left| \lambda_i - \frac{2(3.1)}{5} \right|$$

$= |\lambda_1 - 1.24| + |\lambda_2 - 1.24| + |\lambda_3 - 1.24| + |\lambda_4 - 1.24| + |\lambda_5 - 1.24|$

Substituting $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_5 values and solving, we get

$LE\left(A_\gamma(HG)\right) = 2.6675$

The signless Laplacian energy of SLAM of nonmembership function $A_\gamma(HG)$ is 2.6675

(iii) The signless Laplacian energy of SLAM of hesitant function $A_\beta(HG)$ is

$$A_\beta(HG) = \begin{bmatrix} 0 & 0.2 & 0.2 & 0 & 0.2 \\ 0.2 & 0 & 0.1 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0 & 0.1 & 0 \\ 0 & 0.1 & 0.1 & 0 & 0.1 \\ 0.2 & 0.2 & 0 & 0.1 & 0 \end{bmatrix}$$

The SLAM of hesitant function of HFG is

$$S_L\left(A_\beta(HG)\right) = D_\beta(HG) + A_\beta(HG)$$

By using the SLAM of hesitant function, the degree of the matrix $D_\beta(HG)$ is defined as follows

$$D_\beta(HG) = \begin{bmatrix} 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}$$

Substituting $D_\beta(HG)$ and $A_\beta(HG)$ in the equation of SLAM of hesitant function of HFG, we get

$$S_L\left(A_\beta(HG)\right) = \begin{bmatrix} 0.6 & 0.2 & 0.2 & 0 & 0.2 \\ 0.2 & 0.6 & 0.1 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0.4 & 0.1 & 0 \\ 0 & 0.1 & 0.1 & 0.3 & 0.1 \\ 0.2 & 0.2 & 0 & 0.1 & 0.5 \end{bmatrix}$$

We calculate the eigenvalues of the SLAM $S_L\left(A_\beta(HG)\right)$ of hesitant function by solving its characteristic equation is

$\lambda_1 = 0.1412, \lambda_2 = 0.3277, \lambda_3 = 0.4000, \lambda_4 = 0.4823$, and $\lambda_5 = 1.0488$

The signless Laplacian Energy of SLAM $S_L\left(A_\beta(HG)\right)$ of hesitant function of HFG

$$S_{LE}\left(A_\beta(HG)\right) = \sum_{i=1}^{n} \left| \lambda_i - \frac{2\sum \beta(u_i, u_j)}{n} \right|$$

$$= \sum_{i=1}^{5} \left| \lambda_i - \frac{2(1.2)}{5} \right|$$

$= |\lambda_1 - 0.32| + |\lambda_2 - 0.32| + |\lambda_3 - 0.32| + |\lambda_4 - 0.32| + |\lambda_5 - 0.32|$

Substitute $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_5 values and solving we get

$S_{LE}\left(A_\beta(HG)\right) = 1.1422$

The signless Laplacian energy of SLAM $S_L\left(A_\beta(HG)\right)$ of hesitant function of HFG is 1.1422

The signless Laplacian energy of HFG is (1.8543,2.6675,1.1422)

3. RESULTS

Theorem 3.1: Let $HG = (V,\ E, \mu, \gamma, \beta)$ be a HFG with $|V| = n$ vertices and$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ is theeigenvalues of signless Laplacian adjacency matrix of membership function of hesitancy fuzzy graph then

$$(a) \sum_{i=1}^{n} \lambda_i{}^2 = 2 \sum_{1 \leq i \leq j \leq n} \mu_{ij}$$

$$(b) \sum_{i=1}^{n} \lambda_i{}^2 = 2 \sum_{1 \leq i \leq j \leq n} \mu_{ij}{}^2 + \sum_{i=1}^{n} d_{\mu_{ij}}{}^2 (v_i)$$

Proof: - (a) we know that $A_\mu(HG)$ be the symmetric matrix

Also $S_L\big(A_\mu(HG)\big)$ be the symmetric matrix and the eigenvalues of SLAM$S_L[\mu_{ij}(HG)]$ are non-negative such that

$$tr\Big(L\big(A_\mu(HG)\big)\Big) = \sum_{i=1}^{n} d_{\mu_{ij}(HG)}(v_i) = 2 \sum_{1 \leq i \leq j \leq n} \mu_{ij}$$

Similarly, we get the remaining eigenvalues of SLAM of non-membership function and hesitant function are defined as

$$tr\Big(S_L\big(A_\gamma(HG)\big)\Big) = \sum_{i=1}^{n} d_{\gamma_{ij}(HG)}(v_i) = 2 \sum_{1 \leq i \leq j \leq n} \gamma_{ij}$$

and

$$tr\Big(S_L\big(A_\beta(HG)\big)\Big) = \sum_{i=1}^{n} d_{\beta_{ij}(HG)}(v_i) = 2 \sum_{1 \leq i \leq j \leq n} \beta_{ij}$$

Where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ is the eigenvalues of SLAM of membership function ofHFG.

(b) By the definition of SLAM, we get

$$\begin{bmatrix} d_{\mu_{ij}(HG)}(v_1) & \mu(v_1, v_2) & \dots & \mu(v_1, v_n) \\ \mu(v_2, v_1) & d_{\mu_{ij}(HG)}(v_2) & \vdots & \mu(v_2, v_n) \\ \dots & \dots & & \dots \\ \mu(v_n, v_1) & \mu(v_n, v_2) & \cdots & d_{\mu_{ij}(HG)}(v_n) \end{bmatrix}$$

Now we get $tr\left(S_{L_{(i,i)}}{}^2\right) = \sum_{i=1}^{n} \mu_i{}^2$

Where $tr\left(S_{L_{(i,i)}}{}^2\right) = \left[d^2{}_{\mu_{ij}(HG)}(v_1) + \mu^2(v_1, v_2) + \cdots + \mu^2(v_1, v_n)\right] +$

$\left[\mu^2(v_2, v_1)\ + d^2{}_{\mu_{ij}(HG)}(v_2) + \cdots + \mu^2(v_2, v_n)\right] + \cdots +$

$$[\mu^2(v_n, v_1) + \mu^2(v_n, v_2) + \cdots + d^2{}_{\mu_{ij}(HG)}(v_n)] \sum_{i=1}^{n} \lambda_i{}^2$$

$$= 2 \textstyle\sum_{1 \leq i \leq j \leq n} \mu^2{}_{ij} + \sum_{i=1}^{n} d^2{}_{\mu_{ij}(HG)}(v_i)$$

Similarly, we can prove that for the SLAM of non-membership function and hesitant function are defined as

$$\sum_{i=1}^{n} \lambda_i^{\,2} 2 \sum_{1\le i\le j\le n} \gamma^2{}_{ij} + \sum_{i=1}^{n} d^2{}_{\gamma_{ij}(HG)}(v_i)$$

and

$$\sum_{i=1}^{n} \lambda_i^{\,2} = 2 \sum_{1\le i\le j\le n} \beta^2{}_{ij} + \sum_{i=1}^{n} d^2{}_{\beta_{ij}(HG)}(v_i)$$

Where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$are the eigenvalues of SLAM of membership function of HFG.

Corollary 3.1: Let $HG = (V, E, \mu, \gamma, \beta)$ be a HFG with $|V| = n$ vertices and $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ are the eigenvalues of SLAM of membership function of HFG, where $\Psi_i = \lambda_i - \frac{2\sum_{1\le i\le j\le n}\mu_{ij}{}^2}{n}$then we obtain

$$(a) \sum_{i=1}^{n} \Psi_i = 0 \qquad (b) \sum_{i=1}^{n} \Psi_i^{\,2} = 2M$$

Where $M = \sum_{1\le i\le j\le n} \mu_{ij}{}^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\mu_{ij}(HG)}(v_i) - \frac{2\sum_{1\le i\le j\le n}\mu_{ij}}{n}\right)^2$

Similarly, the eigenvalues of SLAM of nonmembership function of HFG, where
$\chi_i = \delta_i - \frac{2\sum_{1\le i\le j\le n}\gamma_{ij}}{n}$ then we have

$$(a) \sum_{i=1}^{n} \chi_i = 0 \qquad (b) \sum_{i=1}^{n} \chi_i^{\,2} = 2N$$

Where N= $\sum_{1\le i\le j\le n} \gamma_{ij}{}^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\gamma_{ij}(HG)}(v_i) - \frac{2\sum_{1\le i\le j\le n}\gamma_{ij}}{n}\right)^2$

In the same way, the eigenvalues of SLAM of hesitant function of HFG, where
$\Psi_i = \lambda_i - \frac{2\sum_{1\le i\le j\le n}\mu_{ij}{}^2}{n}$ then we obtain

$$(a) \sum_{i=1}^{n} \Psi_i = 0 \qquad (b) \sum_{i=1}^{n} \Psi_i^{\,2} = 2R$$

Where R= $\sum_{1\le i\le j\le n} \beta_{ij}{}^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\beta_{ij}(HG)}(v_i) - \frac{2\sum_{1\le i\le j\le n}\beta_{ij}}{n}\right)^2$

Definition 3.1: Let $HG = (V, E, \mu, \gamma, \beta)$ be HFG with $|V| = n$ vertices and $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ are aneigenvalue of SLAM of membership function of HFG. The signless Laplacian energy of HFG is defined as,

$$S_{LE}(HG) = \left|\lambda_i - \frac{2\sum_{1\le i\le j\le n}\mu(v_i, v_j)}{n}\right|.$$

The Laplacian energy of HFG with the signless Laplacian energy of SLAM of membership function, nonmembership function and hesitant function are defined as

$$S_{LE}(HG) = \left[S_{LE}\left(A_\mu(HG)\right), S_{LE}\left(A_\gamma(HG)\right), S_{LE}\left(A_\beta(HG)\right)\right].$$

Theorem 3.2: Let $HG = (V,\ E, \mu, \gamma, \beta)$ be a HFG with $|V| = n$ and $|E| = m$ and $A(HG) = \left(A_\mu(HG), A_\gamma(HG), A_\beta(HG)\right)$ be a adjacency matrix of HFG and the SLAM of HFG is $S_L\left(A(HG)\right) = \left(S_L\left(A_\mu(HG)\right), S_L\left(A_\gamma(HG)\right), S_L\left(A_\beta(HG)\right)\right)$ then

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ISSN: 2319-9253

$$(i) S_{LE}\left(A_\mu(HG)\right) \leq \sqrt{2n\left(\frac{1}{2}\sum_{i=1}^{n}\left(d_{\mu_{ij}(HG)}(v_i)+\frac{2\sum_{1\leq i\leq j\leq n}\mu_{ij}}{n}\right)^2+\sum_{1\leq i\leq j\leq n}\mu_{ij}{}^2\right)}$$

$$(ii) S_{LE}\left(A_\gamma(HG)\right) \leq \sqrt{2n\left(\frac{1}{2}\sum_{i=1}^{n}\left(d_{\gamma_{ij}(HG)}(v_i)+\frac{2\sum_{1\leq i\leq j\leq n}\gamma_{ij}}{n}\right)^2+\sum_{1\leq i\leq j\leq n}\gamma_{ij}{}^2\right)}$$

$$(iii) S_{LE}\left(A_\beta(HG)\right) \leq \sqrt{2n\left(\frac{1}{2}\sum_{i=1}^{n}\left(d_{\beta_{ij}(HG)}(v_i)+\frac{2\sum_{1\leq i\leq j\leq n}\beta_{ij}}{n}\right)^2+\sum_{1\leq i\leq j\leq n}\beta_{ij}{}^2\right)}$$

Proof: -(i) By the Cauchy –Schwarz inequality condition, we get

$$\left|\sum_{i=1}^{n}\Psi_i\right|^2 \leq n\sum_{i=1}^{n}|\Psi_i|^2$$

Where

$$S_{LE}\left(A_\mu(HG)\right) \leq \sqrt{n\sum_{i=1}^{n}|\Psi_i|^2} \qquad (1)$$

We know that $\sum_{i=1}^{n}|\Psi_i|^2 = 2M$ (2)

By substituting (2) in (1) we get

$$S_{LE}\left(A_\mu(HG)\right) = \sqrt{2Mn}$$

Here $M = \sum_{1\leq i\leq j\leq n}\mu_{ij}{}^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{\mu_{ij}(HG)}(v_i)+\frac{2\sum_{1\leq i\leq j\leq n}\mu_{ij}}{n}\right)^2$

By substituting M value in above equation (1) we get

$$S_{LE}\left(A_\mu(HG)\right) \leq \sqrt{2n\left(\sum_{1\leq i\leq j\leq n}\mu_{ij}{}^2+\frac{1}{2}\sum_{i=1}^{n}\left(d_{\mu_{ij}(HG)}(v_i)+\frac{2\sum_{1\leq i\leq j\leq n}\mu_{ij}}{n}\right)^2\right)}$$

In the same way, we can prove that

$$(ii) S_{LE}\left(A_\gamma(HG)\right) \leq \sqrt{2n\left(\frac{1}{2}\sum_{i=1}^{n}\left(d_{\gamma_{ij}(HG)}(v_i)+\frac{2\sum_{1\leq i\leq j\leq n}\gamma_{ij}}{n}\right)^2+\sum_{1\leq i\leq j\leq n}\gamma_{ij}{}^2\right)}$$

$$(iii) S_{LE}\left(A_\beta(HG)\right) \leq \sqrt{2n\left(\frac{1}{2}\sum_{i=1}^{n}\left(d_{\beta_{ij}(HG)}(v_i)-\frac{2\sum_{1\leq i\leq j\leq n}\beta_{ij}}{n}\right)^2+\sum_{1\leq i\leq j\leq n}\beta_{ij}{}^2\right)}$$

Hence the theorem is proved

Theorem 3.3: - Let $HG = (V, E, \mu, \gamma, \beta)$ be a HFG with vertices $|V| = n$ and $S_L(A(HG)) = \left(S_L(A_\mu(HG)), S_L(A_\gamma(HG)), S_L(A_\beta(HG))\right)$ be the SLAM of HFG, then

$$(i)S_{LE}(A_\mu(HG)) \geq \sqrt{2\left(\sum_{i=1}^{n}\left(d_{\mu_{ij}(HG)}(v_i) + \frac{2\sum_{1\leq i\leq j\leq n}\mu_{ij}}{n}\right)^2 + 2\sum_{1\leq i\leq j\leq n}\mu_{ij}{}^2\right)}$$

$$(ii)S_{LE}(A_\gamma(HG)) \geq \sqrt{2\left(\sum_{i=1}^{n}\left(d_{\gamma_{ij}(HG)}(v_i) + \frac{2\sum_{1\leq i\leq j\leq n}\gamma_{ij}}{n}\right)^2 + 2\sum_{1\leq i\leq j\leq n}\gamma_{ij}{}^2\right)}$$

$$(iii)S_{LE}(A_\beta(HG)) \geq \sqrt{2\left(\sum_{i=1}^{n}\left(d_{\beta_{ij}(HG)}(v_i) + \frac{2\sum_{1\leq i\leq j\leq n}\beta_{ij}}{n}\right)^2 + 2\sum_{1\leq i\leq j\leq n}\beta_{ij}{}^2\right)}$$

Proof: (i) Given $HG = (V,\ E, \mu, \gamma, \beta)$ is a HFG with vertices $|V| = n$ and $S_L(HG) = \left(S_L(A_\mu(HG)), S_L(A_\gamma(HG)),\ S_L(A_\beta(HG))\right)$ is the SLAM of HFG.

Now we have to prove that

$$S_{LE}(A_\mu(HG)) \geq \sqrt{2\left(\sum_{i=1}^{n}\left(d_{\mu_{ij}(HG)}(v_i) + \frac{2\sum_{1\leq i\leq j\leq n}\mu_{ij}}{n}\right)^2 + 2\sum_{1\leq i\leq j\leq n}\mu_{ij}{}^2\right)}$$

By the definition 3.1, we get

$$\left(S_{LE}(A_\mu(HG))\right)^2 = \left(\sum_{i=1}^{n}|\Psi_i|\right)^2$$

$$= \sum_{i=1}^{n}|\Psi_i|^2 + 2\sum_{1\leq i\leq j\leq n}|\Psi_i||\Psi_i| \geq 4M$$

$$\left(S_{LE}(A_\mu(HG))\right)^2 \geq 4M$$

We know that the value of M is

$$M = \frac{1}{2}\sum_{i=1}^{n}\left(d_{\mu_{ij}(HG)}(v_i) + \frac{2\sum_{1\leq i\leq j\leq n}\mu_{ij}}{n}\right)^2 + \sum_{1\leq i\leq j\leq n}\mu_{ij}{}^2$$

Substitute M value in the above equation and calculating, we get

$$S_{LE}(A_\mu(HG)) \geq \sqrt{2\left(\sum_{i=1}^{n}\left(d_{\mu_{ij}(HG)}(v_i) + \frac{2\sum_{1\leq i\leq j\leq n}\mu_{ij}}{n}\right)^2 + 2\sum_{1\leq i\leq j\leq n}\mu_{ij}{}^2\right)}$$

Hence proved the result.

In the same way, we can prove that the remaining results

$$(ii)S_{LE}(A_\gamma(HG)) \geq \sqrt{2\left(\sum_{i=1}^{n}\left(d_{\gamma_{ij}(HG)}(v_i) + \frac{2\sum_{1\leq i\leq j\leq n}\gamma_{ij}}{n}\right)^2 + 2\sum_{1\leq i\leq j\leq n}\gamma_{ij}{}^2\right)}$$

$$(iii) S_{LE}\left(A_{\beta}(HG)\right) \geq \sqrt{2\left(\sum_{i=1}^{n}\left(d_{\beta_{ij}(HG)}(v_i) + \frac{2\sum_{1\leq i\leq j\leq n}\beta_{ij}}{n}\right)^2 + 2\sum_{1\leq i\leq j\leq n}{\beta_{ij}}^2\right)}$$

Theorem 4.3: Let $HG = (V,\ E, \mu, \gamma, \beta)$ be a HFG with vertices $|V| = n$ and $L(HG) = \left(S_L\left(A_\mu(HG)\right),\ S_L\left(A_\gamma(HG)\right),\ S_L\left(A_\beta(HG)\right)\right)$ be the SLAM of HFG, then

$(i) S_{LE}\left(A_\mu(HG)\right) \leq$

$$\Psi_1 + \sqrt{(n-1)\left(\sum_{i=1}^{n}\left(d_{\mu_{ij}(HG)}(v_i) + \frac{2\sum_{1\leq i\leq j\leq n}\mu_{ij}}{n}\right)^2 + 2\left(\sum_{1\leq i\leq j\leq n}{\mu_{ij}}^2\right) - {\Psi_1}^2\right)}$$

$(ii) S_{LE}\left(A_\gamma(HG)\right) \leq$

$$\Psi_1 + \sqrt{(n-1)\left(\sum_{i=1}^{n}\left(d_{\gamma_{ij}(HG)}(v_i) + \frac{2\sum_{1\leq i\leq j\leq n}\gamma_{ij}}{n}\right)^2 + 2\left(\sum_{1\leq i\leq j\leq n}{\gamma_{ij}}^2\right) - {\Psi_1}^2\right)}$$

$(iii) S_{LE}\left(A_\beta(HG)\right) \leq$

$$\Psi_1 + \sqrt{(n-1)\left(\sum_{i=1}^{n}\left(d_{\beta_{ij}(HG)}(v_i) + \frac{2\sum_{1\leq i\leq j\leq n}\gamma_{ij}}{n}\right)^2 + 2\left(\sum_{1\leq i\leq j\leq n}{\beta_{ij}}^2\right) - {\Psi_1}^2\right)}$$

Proof: By Cauchy –Schwarz inequality to (1, 1… 1) and$(|\Psi_1|, |\Psi_1|, \ldots, |\Psi_1|)$, we get

$$\left|\sum_{i=1}^{n}\Psi_i\right|^2 \leq (n-1)(|\Psi_2|^2 + |\Psi_3|^2 + \cdots + |\Psi_n|^2)$$

$$S_{LE}\left(A_\mu(HG)\right) - \Psi_1 \leq \sqrt{(n-1)(2M - {\Psi_1}^2)},$$

Where$S_{LE}\left(A_\mu(HG)\right) - \Psi_1 \leq \sqrt{(n-1)(2M - {\Psi_1}^2)}$,

We know that the value of M is

$$M = \frac{1}{2}\sum_{i=1}^{n}\left(d_{\mu_{ij}(HG)}(v_i) + \frac{2\sum_{1\leq i\leq j\leq n}\mu_{ij}}{n}\right)^2 + \sum_{1\leq i\leq j\leq n}{\mu_{ij}}^2,$$

Thus

$S_{LE}\left(A_\mu(HG)\right) \leq$

$$\Psi_1 + \sqrt{(n-1)\left(\sum_{i=1}^{n}\left(d_{\mu_{ij}(HG)}(v_i) + \frac{2\sum_{1\leq i\leq j\leq n}\mu_{ij}}{n}\right)^2 + 2\left(\sum_{1\leq i\leq j\leq n}{\mu_{ij}}^2\right) - {\Psi_1}^2\right)}$$

Hence the proof is completed.

In the same way, we can prove that the remaining two conditions.

$(ii) S_{LE}\left(A_\gamma(HG)\right) \leq$

$$\Psi_1 + \sqrt{(n-1)\left(\sum_{i=1}^{n}\left(d_{\gamma_{ij}(HG)}(v_i) + \frac{2\sum_{1\le i\le j\le n}\gamma_{ij}}{n}\right)^2 + 2\left(\sum_{1\le i\le j\le n}\gamma_{ij}^2\right) - \Psi_1^2\right)}$$

$(iii) S_{LE}(A_\beta(HG)) \le$

$$\Psi_1 + \sqrt{(n-1)\left(\sum_{i=1}^{n}\left(d_{\beta_{ij}(HG)}(v_i) + \frac{2\sum_{1\le i\le j\le n}\beta_{ij}}{n}\right)^2 + 2\left(\sum_{1\le i\le j\le n}\beta_{ij}^2\right) - \Psi_1^2\right)}$$

4. CONCLUSION

This research derived upper and lower bounds for the signless Laplacian energy based on important graph parameters such as degree and eigenvalues. Furthermore, we provided an illustrative example to demonstrate the validity of the results. The findings of this research contribute to the deeper understanding of properties in hesitancy fuzzy graph theory and the future applications in areas such as decision-making, artificial intelligence, and network analysis.

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